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On a Hopf hypersurface of a complex space form

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We prove that if the sectional curvatures for plane sections containing the structure vector field of a real hypersurface in a complex space form are equal to the same constant at every point, then the real hypersurface is a Hopf hypersurface. We also find some applications of the main result.

1. Introduction

Let $M^n(c)$ be a complex n-dimensional complex space form with constant holomorphic sectional curvature $4c$, and let M be a real hypersurface of $M^n(c)$. We denote by J the complex structure of $M^n(c)$. Then M has an induced almost contact metric structure (ϕ, ξ, η, g). When the structure vector field ξ is a principal curvature vector field, we call M a Hopf hypersurface.

We define the holomorphic distribution T_0 on M by $T_0(x) = \{X \in T_x(M) \mid \eta(X) = 0\}$. It is an interesting and important problem to determine real hypersurfaces of complex space forms with respect to some conditions on the holomorphic distribution on real hypersurfaces. For instance, Kimura [3] classified real hypersurfaces of a complex space form CP^n, $n \geqslant 3$, on which the sectional curvature of the holomorphic 2-plane spanned by a unit tangent vector orthogonal to the structure vector field ξ is constant.

In this paper we prove that if the sectional curvature K for plane sections containing ξ is a constant k at every point of M, then M is congruent to a Hopf hypersurface which satisfies $A\xi = 0$ $(k = c)$ or a totally η-umbilical real hypersurface $(k \neq c)$.

Concerning the above condition, the following result is well known: in order for a $(2n+1)$-dimensional Riemannian manifold M to be a K-contact metric manifold, it is necessary and sufficient that M would admit a unit Killing vector field ξ and that the sectional curvature for plane sections containing ξ would be equal to 1 at every point of M (cf. [1]).

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2. Preliminaries

Let $M^n(c)$ denote the complex space form of complex dimension n (real dimension $2n$) with constant holomorphic sectional curvature $4c$. We denote by J the complex structure of $M^n(c)$. The Hermitian metric of $M^n(c)$ will be denoted by G.

Let M be a real $(2n-1)$-dimensional hypersurface immersed in $M^n(c)$. We denote by g the Riemannian metric induced on M from G. We take the unit normal vector field V of M in $M^n(c)$. For any vector field X tangent to M, we define ϕ, η and ξ by

$$JX = \phi X + \eta(X)V, \qquad JV = -\xi,$$

where ϕX is the tangential part of JX, ϕ is a tensor field of type $(1,1)$, η is a 1-form, and ξ is the unit vector field on M. Then they satisfy

$$\begin{aligned}
&\phi^2 X = -X + \eta(X)\xi, \qquad \phi\xi = 0, \qquad \eta(\phi X) = 0,\\
&g(\phi X, Y) + g(X, \phi Y) = 0, \qquad \eta(X) = g(X, \xi),\\
&g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).
\end{aligned}$$

Thus (ϕ, ξ, η, g) defines an almost contact metric structure on M.

We denote by $\tilde{\nabla}$ the operator of covariant differentiation in $M^n(c)$, and by ∇ the one in M determined by the induced metric. Then the *Gauss and Weingarten formulas* are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)V, \qquad \tilde{\nabla}_X V = -AX$$

for any vector fields X and Y tangent to M. We call A the *shape operator* of M.

For the almost contact metric structure on M, we have

$$\nabla_X \xi = \phi AX, \qquad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi.$$

We denote by R the Riemannian curvature tensor field of M. Then the *equation of Gauss* is given by

$$\begin{aligned}
R(X,Y)Z = {} & c\big(g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\big)\\
& + g(AY, Z)AX - g(AX, Z)AY,
\end{aligned}$$

and the *equation of Codazzi* by

$$(\nabla_X A)Y - (\nabla_Y A)X = c\big(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\big).$$

If the shape operator A of M is of the form $AX = aX + b\eta(X)\xi$ for some functions a and b, then M is said to be *totally η-umbilical* (see Tashiro and Tachibana [7]). It is well known that if M is a totally η-umbilical real hypersurface of a complex space form $M^n(c)$, $c \neq 0$, $n \geqslant 2$, then M has two constant principal curvatures (see Takagi [6]). When M is totally η-umbilical, then the sectional curvature $K(X,\xi) = g(R(X,\xi)\xi, X) = c + a^2 + ab - b^2$ is constant for any unit vector X orthogonal to ξ.

We put $\alpha = g(A\xi, \xi)$. If ξ is a principal vector everywhere (i.e. $A\xi = \alpha\xi$), we say that M is a *Hopf hypersurface*. Any totally η-umbilical real hypersurface is a Hopf hypersurface.

We use the following theorems (see Cecil and Ryan [2] and Montiel [5]):

Theorem A. *Let M be a real hypersurface of CP^n, $n \geqslant 3$, with 2 distinct principal curvatures. Then M is locally congruent to a geodesic hypersphere.*

Theorem B. *Let M be a real hypersurface of CH^n, $n \geqslant 3$, with 2 distinct principal curvatures. Then M is locally congruent to one of the following spaces*:

(a) *a geodesic hypersphere,*
(b) *a tube over a complex hyperbolic hyperplane,*
(c) *a horosphere,*
(d) *a tube of radius* $\log((1+\sqrt{3})/\sqrt{2})$ *over a totally real hyperbolic hyperplane.*

3. Hopf hypersurfaces

The main purpose of this section is to prove the following

Theorem 1. *Let M be a real hypersurface of a complex space form $M^n(c)$, $n \geqslant 3$, $c \neq 0$. If the sectional curvature K for plane sections containing ξ is a constant k at every point of M, then M is a Hopf hypersurface.*

Let M be a real hypersurface of $M^n(c)$, $n \geqslant 3$, $c \neq 0$. For any vector field X orthogonal to ξ, the sectional curvature K of M satisfies

$$K(X,\xi)g(X,X) = g\big(R(X,\xi)\xi, X\big) = cg(X,X) + g(A\xi,\xi)g(AX,X) - g(AX,\xi)^2.$$

We put $\alpha = g(A\xi,\xi)$. Suppose that the sectional curvature K of M for plane sections containing ξ is a constant k at every point of M. Then

$$(k-c)g(X,X) = \alpha g(AX,X) - g(AX,\xi)^2$$

for any vector field X orthogonal to ξ. From this we have

$$(k-c)g(X,Y) = \alpha g(AX,Y) - g(AX,\xi)g(AY,\xi) \tag{1}$$

for any vector fields X and Y orthogonal to ξ.

Assuming that M is not Hopf, we can find a point where $A\xi \neq \alpha\xi$. Hence on a sufficiently small neighborhood $\mathcal{N}$ of that point, we put $A\xi = \alpha\xi + hU$, where U is a unit vector field orthogonal to ξ, α and h are differentiable functions. Moreover, we put $\lambda = g(AU,U)$. Then

$$k - c = \alpha\lambda - h^2, \tag{2}$$

$$k - c = \alpha g(AX,X) \tag{3}$$

for any vector field X orthogonal to ξ and U. We note that α cannot vanish since this would imply $k = c$ and $h = 0$. Thus, for any vector field X orthogonal to ξ and U, we have

$$AX = \beta X,$$

where $\beta = (k-c)/\alpha$.

Lemma 2. *For any vector fields X and Y orthogonal to ξ and U, we have*

$$h\beta g(\phi X,Y) + (\lambda-\beta)g(\nabla_X U,Y) - (U\beta)g(X,Y) = 0, \tag{4}$$

$$\big(c + \alpha\beta - \beta^2\big)g(\phi X,Y) + hg(\nabla_X U,Y) - (\xi\beta)g(X,Y) = 0, \tag{5}$$

$$(2c - 2\lambda\beta + \alpha\beta + \lambda\alpha)g(\phi X,U) + hg(\nabla_U X,U) + (Xh) = 0, \tag{6}$$

$$h(2\beta+\lambda)g(\phi X,U) + (\lambda-\beta)g(\nabla_U X,U) + (X\lambda) = 0. \tag{7}$$

Proof. By the equation of Codazzi,

$$g\big((\nabla_X A)U - (\nabla_U A)X, Y\big) = 0.$$

On the other hand, using $\nabla_X\xi = \phi AX$,

$$\begin{aligned} g\big((\nabla_X A)U - (\nabla_U A)X, Y\big) &= g\big(\nabla_X(AU) - A\nabla_X U - \nabla_U(AX) + A\nabla_U X, Y\big) \\ &= g\big(\nabla_X(\lambda U + h\xi), Y\big) - g(\nabla_X U, \beta Y) - g\big(\nabla_U(\beta X), Y\big) + g(\nabla_U X, \beta Y) \\ &= h\beta g(\phi X,Y) + (\lambda-\beta)g(\nabla_X U,Y) - (U\beta)g(X,Y). \end{aligned}$$

Thus we obtain (4). Next, by the equation of Codazzi,

$$g\big((\nabla_X A)\xi - (\nabla_\xi A)X, Y\big) = -cg(\phi X,Y).$$

On the other hand, we have

$$\begin{aligned} g\big((\nabla_X A)\xi - (\nabla_\xi A)X, Y\big) &= g\big(\nabla_X(A\xi) - A\nabla_X\xi - \nabla_\xi(AX) + A\nabla_\xi X, Y\big) \\ &= \big(\alpha\beta - \beta^2\big)g(\phi X,Y) + hg(\nabla_X U,Y) - (\xi\beta)g(X,Y). \end{aligned}$$

This proves (5). The equation of Codazzi implies

$$g\big((\nabla_X A)U - (\nabla_U A)X, \xi\big) = -2cg(\phi X,U).$$

On the other hand, we see that

$$\begin{aligned} g\big((\nabla_X A)U - (\nabla_U A)X, \xi\big) &= g\big(\nabla_X(AU) - A\nabla_X U - \nabla_U(AX) + A\nabla_U X, \xi\big) \\ &= \lambda g(\nabla_X U,\xi) + (Xh) - g(\nabla_X U, A\xi) - \beta g(\nabla_U X,\xi) + g(\nabla_U X, A\xi) \\ &= -\lambda g(U,\phi AX) + (Xh) + \alpha g(U,\phi AX) + \beta g(X,\phi AU) + hg(\nabla_U X,U) - \alpha g(X,\phi AU) \\ &= (-2\lambda\beta + \alpha\beta + \lambda\alpha)g(\phi X,U) + hg(\nabla_U X,U) + (Xh). \end{aligned}$$

Thus we have (6). By the similar computation, we obtain

$$g\big((\nabla_X A)U - (\nabla_U A)X, U\big) = h(2\beta + \lambda)g(\phi X, U) + (\lambda - \beta)g(\nabla_U X, U) + (X\lambda).$$

Moreover, from the equation of Codazzi,

$$g\big((\nabla_X A)U - (\nabla_U A)X, U\big) = 0.$$

This shows (7). □

There exist some X, Y that satisfy $g(X, Y) = 0$ and $g(\phi X, Y) \neq 0$. From (4) and (5), we get

$$\big\{\beta h^2 - (\lambda - \beta)\big(c + \alpha\beta - \beta^2\big)\big\}g(\phi X, Y) = 0.$$

From (2) and (3), we have $h^2 = \lambda\alpha - \alpha\beta$. Using these equations,

$$\big(\beta^2 - c\big)(\lambda - \beta) = 0.$$

We can observe that $c > 0$ since $\lambda \neq \beta$. From (3) and $\beta^2 = c$, we see that $\alpha = (k - c)/\beta$ is constant. Using (6) and (7), we have

$$(4hc - 4\lambda\beta h + \lambda\alpha h)g(\phi X, U) + \big(2h^2 - \lambda\alpha + \alpha\beta\big)g(\nabla_U X, U) + 2h(Xh) - \alpha(X\lambda) = 0.$$

From (2), we have $2h(Xh) - \alpha(X\lambda) = 0$. Thus

$$(4c - 4\lambda\beta + \lambda\alpha)g(\phi X, U) + hg(\nabla_U X, U) = 0. \tag{8}$$

From this equation and Lemma 2, we have

Lemma 3. *For any vector field X orthogonal to ξ, U, and ϕU, we have*

$$\nabla_X U = \frac{h\beta}{\beta - \lambda}\phi X, \qquad \nabla_{\phi X} U = \frac{-h\beta}{\beta - \lambda}X,$$
$$\nabla_U U = \frac{-4c + 4\lambda\beta - \lambda\alpha}{h}\phi U.$$

Proof. Since $\beta^2 = c$ is constant, (4) implies

$$g(\nabla_X U, Y) = \frac{h\beta}{\beta - \lambda}g(\phi X, Y)$$

for any vector field Y orthogonal to ξ and U. On the other hand, we have

$$g(\nabla_X U, \xi) = -g(U, \phi AX) = \beta g(\phi U, X) = 0.$$

Hence we obtain the first equation. Next we compute $\nabla_{\phi X} U$. By (4),

$$g(\nabla_{\phi X} U, Y) = \frac{h\beta}{\beta - \lambda}g\big(\phi^2 X, Y\big) = -\frac{h\beta}{\beta - \lambda}g(X, Y).$$

On the other hand, since $A\phi X = \beta\phi X$ for any X orthogonal to ξ, U, ϕU, we have

$$g(\nabla_{\phi X} U, \xi) = -g(U, \phi A\phi X) = -\beta g\big(U, \phi^2 X\big) = 0.$$

These equations imply the second equation.

Using (8), we see that

$$g(\nabla_U U, X) = \frac{-4c + 4\lambda\beta - \lambda\alpha}{h}g(\phi U, X).$$

Moreover, we have

$$g(\nabla_U U, \xi) = g(U, \phi AU) = 0,$$

from which we obtain the last equation. □

Let X be a unit vector field orthogonal to ξ, U and ϕU. Then, from Lemma 3, the sectional curvature $K(U, X) = g(R(U, X)X, U)$ is given by

$$g\big(R(U, X)X, U\big) = -g(\nabla_X X, \nabla_U U) + g(\nabla_U X, \nabla_X U) - g(\nabla_{[U,X]} X, U).$$

We compute $K(U,X)$. Noticing $(\nabla_X\phi)U=\eta(U)AX-g(AX,U)\xi=0$,

$$-g(\nabla_X X,\nabla_U U)=\frac{4c-4\lambda\beta+\lambda\alpha}{h}g(\nabla_X X,\phi U)=\frac{\beta(4c-4\lambda\beta+\lambda\alpha)}{\beta-\lambda},$$

$$g(\nabla_U X,\nabla_X U)=\frac{h\beta}{\beta-\lambda}g(\nabla_U X,\phi X).$$

Next we compute $g(\nabla_{[U,X]}X,U)$. Using the fact that X is perpendicular to U, ϕU and ξ, we have

$$g(\nabla_U X,\xi)=-g(X,\nabla_U\xi)=-g(X,\phi AU)=0,$$
$$g(\nabla_U X,U)=-g(X,\nabla_U U)=0,$$
$$g(\nabla_U X,\phi U)=-g(X,\phi\nabla_U U)=0.$$

Thus Lemma 3 implies that

$$\nabla_{\nabla_U X}U=\frac{h\beta}{\beta-\lambda}\phi\nabla_U X.$$

Thus we obtain

$$\begin{aligned}-g(\nabla_{[U,X]}X,U)&=-g(\nabla_{\nabla_U X}X,U)+g(\nabla_{\nabla_X U}X,U)\\&=-\frac{h\beta}{\beta-\lambda}g(\nabla_U X,\phi X)-\frac{h\beta}{\beta-\lambda}g(X,\nabla_{\phi X}U)\\&=-\frac{h\beta}{\beta-\lambda}g(\nabla_U X,\phi X)+\frac{h^2\beta^2}{(\beta-\lambda)^2}.\end{aligned}$$

Consequently, again by Lemma 3,

$$g\big(R(U,X)X,U\big)=\frac{\beta(4c-4\lambda\beta+\lambda\alpha)}{\beta-\lambda}+\frac{h^2\beta^2}{(\beta-\lambda)^2}.$$

Since $h^2=\lambda\alpha-\alpha\beta$, it follows that

$$g\big(R(U,X)X,U\big)=\frac{4\beta c-4\lambda\beta^2+\lambda\alpha\beta-\alpha\beta^2}{\beta-\lambda}.$$

On the other hand, by the equation of Gauss,

$$g\big(R(U,X)X,U\big)=c+\lambda\beta.$$

From these equations and $\beta^2=c$, we obtain

$$\begin{aligned}0&=4\beta c-4\lambda\beta^2+\lambda\alpha\beta-\alpha\beta^2-(c+\lambda\beta)(\beta-\lambda)\\&=3\beta^3-4\lambda\beta^2+\lambda\alpha\beta-\alpha\beta^2+\lambda^2\beta\\&=\beta(\lambda-\beta)(\lambda+\alpha-3\beta).\end{aligned}$$

Since $\lambda\neq\beta$ and $\beta\neq 0$, we have $\lambda=-\alpha+3\beta$. So λ and h are constant. Therefore, by (7),

$$g(\nabla_U U,U)=\frac{h(2\beta+\lambda)}{\beta-\lambda}.$$

From (8), we have

$$\frac{h(2\beta+\lambda)}{\beta-\lambda}=\frac{-4c+4\lambda\beta-\lambda\alpha}{h}.$$

Using $h^2=\alpha(\lambda-\beta)$, $\beta^2=c$ and $\lambda=-\alpha+3\beta$,

$$0=-\alpha(2\beta+\lambda)+4c-4\lambda\beta+\lambda\alpha=2\beta(-4\beta+\alpha),$$

from which we obtain $\alpha=4\beta$, and hence $\lambda=-\beta$. Thus we have

$$h^2=\lambda\alpha-\alpha\beta=-8\beta^2\leqslant 0.$$

This is a contradiction. Hence we have our theorem.

Using this theorem, we have the following

Corollary 4. *Let M be a real hypersurface of a complex space form $M^n(c)$, $n \geqslant 3$, $c \neq 0$. If the sectional curvature K for plane sections containing ξ is a constant k at every point of M, then M is locally congruent to one of the following spaces*:

(a) *a Hopf hypersurface which satisfies $A\xi = 0$ $(k = c)$,*
(b) *a totally η-umbilical real hypersurface $(k \neq c)$.*

Proof. By Theorem 1, we see that M is a Hopf hypersurface. Then (1) reduces to

$$\alpha AX = (k - c)X$$

for all X orthogonal to ξ. Since M is a Hopf hypersurface, α is constant. Therefore, if $\alpha = 0$, that is, $A\xi = 0$, then $k = c$. On the other hand, if $\alpha \neq 0$, M has two constant principal curvatures with multiplicities 1 and $2n - 2$, respectively, that is, M is totally η-umbilic.

Conversely, by the direct computation using the equation of Gauss, if the shape operator A satisfies $A\xi = 0$, then the sectional curvature K satisfies $K(X, \xi) = c$ for any unit vector $X \in T_0(x)$ at every point x. Moreover, if M is totally η-umbilic, then the sectional curvature K satisfies $K(X, \xi) = k$, k being a constant, for any unit vector $X \in T_0(x)$ at every point x. So we have our result. □

Remark 1. Let us remind that each totally η-umbilical real hypersurface of a complex space form has two distinct principal curvatures (see [6]). The converse does not hold. For example, all hypersurfaces quoted in Theorem B have two distinct principal curvatures. Here, the hypersurfaces of types (a)–(c) are totally η-umbilical but those of type (d) are not (see [5]).

From Corollary 4, Remark 1, Theorems A and B (see also [4–6]), we have

Theorem 5. *Let M be a real hypersurface of a complex projective space CP^n, $n \geqslant 3$. If the sectional curvature K for plane sections containing ξ is a constant k at every point of M, then M is locally congruent to one of the following spaces*:

(a) *a Hopf hypersurface which satisfies $A\xi = 0$ $(k = 1)$,*
(b) *a geodesic hypersphere of radius θ $(k = t \neq 1)$,*

where we have put $t = \cot^2 \theta$.

Theorem 6. *Let M be a real hypersurface of a complex hyperbolic space CH^n, $n \geqslant 3$. If the sectional curvature K for plane sections containing ξ is a constant k at every point of M, then M is locally congruent to one of the following spaces*:

(a) *a Hopf hypersurface which satisfies $A\xi = 0$ $(k = -1)$,*
(b) *a geodesic hypersphere of radius $\theta > 0$ $(k = 1/t)$,*
(c) *a tube of radius $\theta > 0$ over a complex hyperbolic hyperplane $(k = t)$,*
(d) *a horosphere $(k = 1)$,*

where we have put $t = \tanh^2 \theta$.

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